Failure is Not an Option

The Curry-Howard-Shadok correspondence

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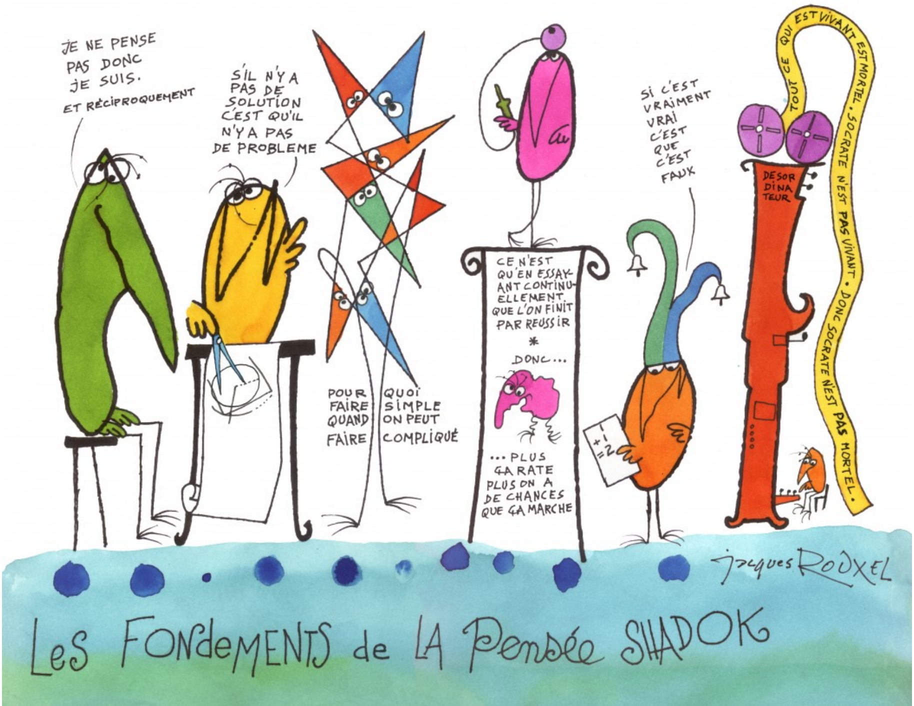
joint work with **Nicolas Tabareau**

Max Planck Institute for Software Systems

Séminaire PPS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QUOI
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*
DONC...
...PLUS ÇA RATE PLUS ON A DE CHANCES QUE ÇA MARCHE
SI C'EST VRAIMENT VRAI C'EST QUE C'EST FAUX
DESORDINATEUR
TOUT CE QUI EST VIVANT EST MORTEL. SOCRATE N'EST PAS VIVANT. DONC SOCRATE N'EST PAS MORTEL.
Jacques ROUXEL
Les FONdeMENTS de LA Pensée SHADOK

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The Pinnacle of the Curry-Howard correspondence

Un Coq qui fait de l'effet

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To *Prove* More!

- A well-known fact here at PPS
- Curry-Howard $\vdash$ side-effects $\Leftrightarrow$ new axioms
- Archetypical example: `callcc` and classical logic (Griffin, Krivine)

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- Computationally: a glorified monotonous reader monad

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Weaning (LICS 2017)

- A generic construction adding effects
- Handles a rather wide class of monads
- Somehow dual to forcing

You Can't Have Your Cake and Eat It

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Call-by-value	Call-by-name
	
☹ Weaker conversion rule	☺ Full conversion rule
☺ Full dependent elimination	☹ Weaker dependent elimination
☺ Good old ML semantics	☹ Strange PL realm

Last Propaganda Slide: A Flurry of Buzzwords

Recall that dependent elimination for booleans amounts to

$$\frac{\Gamma \vdash M : \mathbb{B} \qquad \Gamma \vdash N_1 : P\{\texttt{true}\} \qquad \Gamma \vdash N_2 : P\{\texttt{false}\}}{\Gamma \vdash \texttt{if } M \texttt{ then } N_1 \texttt{ else } N_2 : P\{M\}}$$

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We proposed a generic restriction for effectful CBN dependent elimination.

P **must be linear** ($\cong$ CBV / algebra hom.)

- Generalizes Krivine's storage operators
- If you weren't at my Geocal-LAC talk, *tant pis pour vous*
- Towards a Linear Dependent {Big Data, Machine Learning, IoT}

Shameless Propaganda

Part I

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where $P : \mathcal{I} \to \square$.

Remark that in call-by-name, if $M : A \to B$, in general

$$M\ (\mathtt{raise}\ A\ e) \quad \not\equiv \quad \mathtt{raise}\ B\ e$$

for otherwise we would not have $(\lambda x : A.\, M)\ N \equiv M\{x := N\}$.

Catch Me If You Can

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For inductive types, this is a **generalized induction principle**.

$$\begin{array}{ll} \texttt{catch}_{\mathbb{B}} : & \Pi P : \mathbb{B} \to \square. \\ & \quad P\ \texttt{true} \to \\ & \quad P\ \texttt{false} \to \\ & \quad (\Pi e : \mathbf{E}.\, P\ (\texttt{raise}\ \mathbb{B}\ e)) \to \\ & \Pi b : \mathbb{B}.\, P\ b \end{array} \qquad \begin{array}{ll} \mathbb{B}_{\texttt{rect}} : & \Pi P : \mathbb{B} \to \square. \\ & \quad P\ \texttt{true} \to \\ & \quad P\ \texttt{false} \to \\ & \\ & \Pi b : \mathbb{B}.\, P\ b \end{array}$$

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- What about normalization? Subject reduction? Other nice properties?
- … that's called a model.

We want a **model of** the exceptional type theory!

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Categorical models: abstract description of type theory.

- **Pro:** Abstract, subsumes the two former ones.
- **Con:** Realizability + very low level, gazillion variants, intrisically typed, static.

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On the **logic** side, extend expressivity through proof translation.

- Double-negation $\Rightarrow$ classical logic (`callcc`)
- Friedman's trick $\Rightarrow$ Markov's rule (exceptions)
- Forcing $\Rightarrow$ $\neg\mathrm{CH}$ (global monotonous cell)

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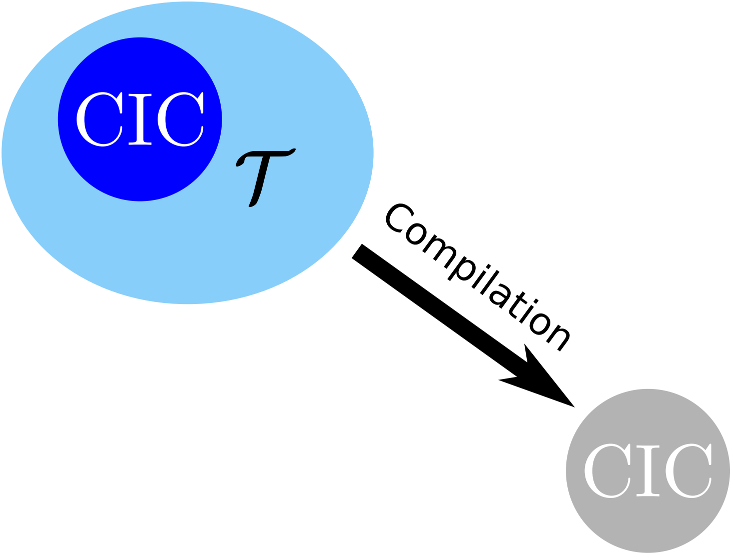
Step 3: Expand $\mathcal{T}$ by going down to the CIC assembly language, implementing new terms given by the $[\cdot]$ translation.



« CIC, the LLVM of Type Theory »

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Only parameter of the translation: a fixed type of exceptions in the target.

$$\vdash_{\mathrm{CIC}} \mathbb{E} : \square$$

The Exceptional Implementation, Negative case

Intuition: $\vdash_{\mathcal{T}_{\mathbb{E}}} A : \square \quad \rightsquigarrow \quad \vdash_{\mathrm{CIC}} [A] : \Sigma A : \square.\, \mathbb{E} \to A.$

Every exceptional type comes with its own implementation of failure!

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[x] & \equiv & x \\
[M\ N] & \equiv & [M]\ [N] \\
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If $\Gamma \vdash_{\mathrm{CIC}} M : A$ then $[\![\Gamma]\!] \vdash_{\mathrm{CIC}} [M] : [\![A]\!]$.

The Exceptional Implementation, Failure

It is straightforward to implement the failure operation.

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Computational rules trivially hold!

$$\begin{array}{ccc} [\mathtt{raise}\ (\Pi x : A.\, B)\ e] & \equiv & [\lambda x : A.\, \mathtt{raise}\ B\ e] \\ ||| & & ||| \\ \pi_2\ ((\Pi x : [\![A]\!].\, [\![B]\!]), (\lambda (e : \mathbb{E})\, (x : [\![A]\!]).\, \pi_2\ [\![B]\!]\ e))\ [e] & \equiv & \lambda x : [\![A]\!].\, \pi_2\ [B]\ [e] \end{array}$$

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The really interesting case is the inductive part of CIC.

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Worse, what about $[\bot]_\varnothing : \mathbb{E} \to [\![\bot]\!]$?

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Very elegant solution: add a default case to every inductive type!

$$\texttt{Inductive}\ [\![\mathbb{B}]\!] := [\texttt{true}] : [\![\mathbb{B}]\!] \mid [\texttt{false}] : [\![\mathbb{B}]\!] \mid \mathbb{B}_\varnothing : \mathbb{E} \to [\![\mathbb{B}]\!]$$

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Pattern-matching is translated pointwise, except for the new case.

$$[\![\Pi P : \mathbb{B} \to \square.\, P\ \texttt{true} \to P\ \texttt{false} \to \Pi b : \mathbb{B}.\, P\ b]\!]$$

$$\equiv \quad \Pi P : [\![\mathbb{B}]\!] \to [\![\square]\!].\, P\ [\texttt{true}] \to P\ [\texttt{false}] \to \Pi b : [\![\mathbb{B}]\!].\, P\ b$$

- If b is $[\texttt{true}]$, use first hypothesis
- If b is $[\texttt{false}]$, use second hypothesis
- If b is an error $\mathbb{B}_\varnothing\ e$, **reraise** e using $[P\ b]_\varnothing\ e$

Shadok Logic Strikes Back

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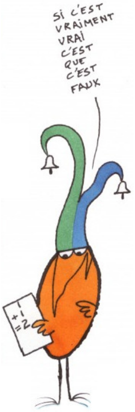

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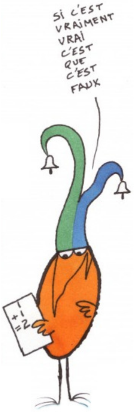
- ☹ Ah, yeah, and also, the theory is inconsistent.

It suffices to raise an exception to inhabit any type.



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And now for a little ad before the second part of the show!

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First-order purification

If P is a Σ^0_1 type, then $\vdash_{\mathrm{CIC}} [\![P]\!] \leftrightarrow P + \mathbb{E}$.

Informercial — Did You Know?

The exceptional translation is just a principled Friedman's A-translation!

As such, it can be used for classical proof extraction.

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Friedman's Trick in CIC

If P and Q are Σ^0_1 types, $\vdash_{\mathrm{CIC}} \Pi p : P.\ \neg\neg Q$ implies $\vdash_{\mathrm{CIC}} \Pi p : P.\ Q$.

Part II

If You Joined the Talk Recently

The exceptional type theory is logically inconsistent!

Cliffhanger (cont.)

You can prove that a program does not raise uncaught exceptions.

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Let's call **valid** a program in $\mathcal{T}_{\mathbb{E}}$ that "does not raise exceptions".

For instance,

- there is no valid proof of $\bot$
- the only valid booleans are `true` and `false`
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Validity is a type-directed notion!

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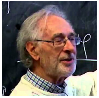

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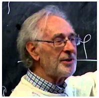

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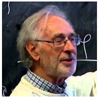

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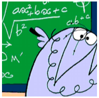

Zo!

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We just have to adapt it to our exceptional translation.

Idea:

From $\vdash M : A$ produce **two** sequents $\begin{cases} \vdash_{\mathrm{CIC}} [M] : [\![A]\!] \\ \quad + \\ \vdash_{\mathrm{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon\ [M] \end{cases}$

where $[\![A]\!]_\varepsilon : [\![A]\!] \to \square$ is the validity predicate.

Parametric Exceptional Translation (Sketch)

Most notably,

$$\begin{array}{lcl}
[\![\Pi x : A.\, B]\!]_\varepsilon\ f & \equiv & \Pi(x : [\![A]\!])\,(x_\varepsilon : [\![A]\!]_\varepsilon\ x).\,[\![B]\!]_\varepsilon\ (f\,x) \\
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Every pure term is now automatically parametric.

If $\Gamma \vdash_{\mathrm{CIC}} M : A$ then $[\![\Gamma]\!]_\varepsilon \vdash_{\mathrm{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon\ [M]$.

A Few Nice Results

Let's call $\mathcal{T}_{\mathbb{E}}^{p}$ the resulting theory. It inherits a lot from CIC!

Theorem (Consistency)

$\mathcal{T}_{\mathbb{E}}^{p}$ is consistent.

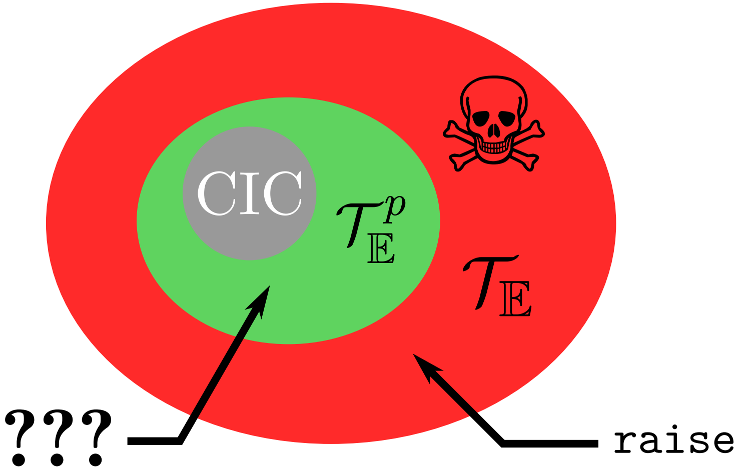
Theorem (Canonicity)

$\mathcal{T}_{\mathbb{E}}^{p}$ enjoys canonicity, i.e if $\vdash_{\mathcal{T}_{\mathbb{E}}^{p}} M : \mathbb{N}$ then $M \rightsquigarrow^{*} \bar{n} \in \bar{\mathbb{N}}$.

Theorem (Syntax)

$\mathcal{T}_{\mathbb{E}}^{p}$ has decidable type-checking, strong normalization and whatnot.

What If There Were No Cake?



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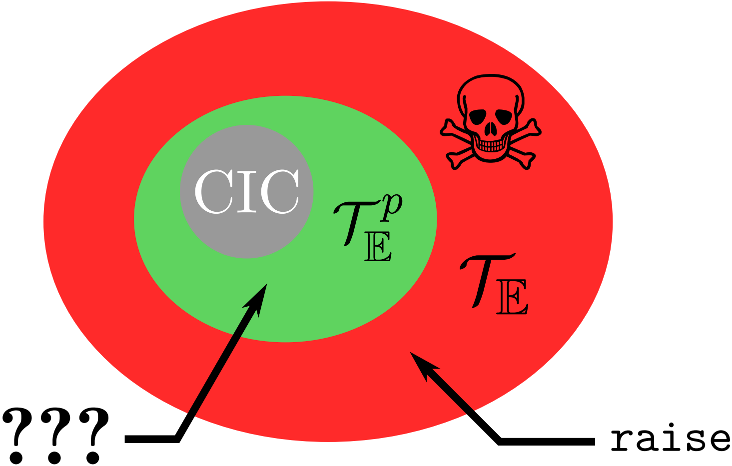

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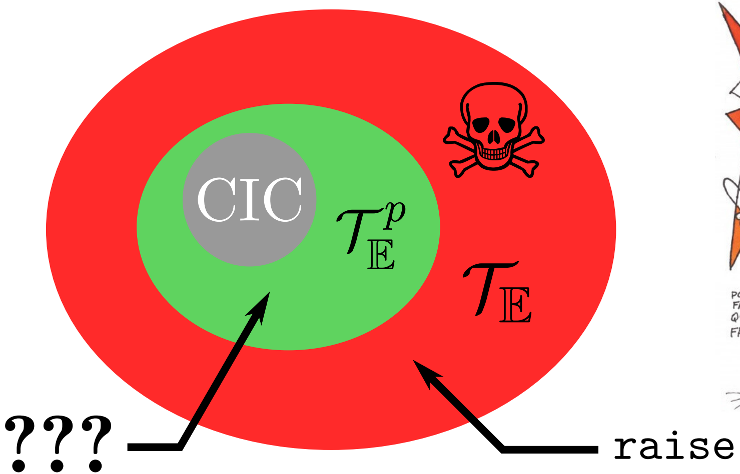

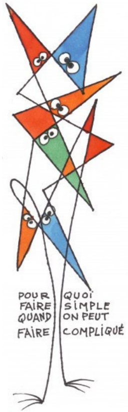

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$\mathcal{T}_{\mathbb{E}}$ is the `unsafe` Coq fragment, and $\mathcal{T}_{\mathbb{E}}^{p}$ a semantical layer atop of it.

Actually $\mathcal{T}_{\mathbb{E}}^{p}$ is the embodiement of Kreisel modified realizability in CIC.

Explaining the Analogy

	Kreisel realizability	$\mathcal{T}^p_{\mathbb{E}}$
Source theory	HA or HA^ω	CIC
Programming language	System T	$\mathcal{T}_{\mathbb{E}}$ ("unsafe Coq")
Logical meta-theory	HA^ω	CIC

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Kreisel realizability extends arithmetic with essentially two principles.

- $\mathrm{AC}_{\mathbb{N}} : (\forall n : \mathbb{N}.\, \exists m : \mathbb{N}.\, P\ (m, n)) \to \exists f : \mathbb{N} \to \mathbb{N}.\, \forall n : \mathbb{N}.\, P\ (n, f\ n)$
- $\mathrm{IP} : (\neg A \to \exists n : \mathbb{N}.\, P\ n) \to \exists n : \mathbb{N}.\, \neg A \to P\ n$

Choice

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Not much to say here.

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In both cases, choice is built-in and a consequence of canonicity.

Independence of Premises

$$\mathrm{IP} : (\neg A \to \exists n : \mathbb{N}.\, P\ n) \to \exists n : \mathbb{N}.\, \neg A \to P\ n$$

That one is interesting! A unforeseen consequence of a subtle **bug**.

Kreisel's bug

Every type of realizers is inhabited. In particular, $[\![\bot]\!]_{\mathrm{KR}} \equiv \mathbb{N}$.

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The realizer of IP critically relies on that!

Assuming System T had an empty type $\mathbb{0}$, and setting $[\![\bot]\!]_{\mathrm{KR}} \equiv \mathbb{0}$

- KR is still a model of HA
- KR still validates $\mathrm{AC}_{\mathbb{N}}$
- KR **doesn't** validate IP anymore

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Theorem $(\mathrm{CIC} + \mathrm{IP})$

$\mathcal{T}_{\mathbb{E}}^{p}$ *validates* IP*, owing to the fact that in* $\mathcal{T}_{\mathbb{E}}$*, every type is inhabited.*

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Proof (sketch).

In $\mathcal{T}_{\mathbb{E}}$, build a term `ip` : IP

- Given $f : \neg A \to \Sigma n : \mathbb{N}.\, P\ n$, apply it to `raise` $(\neg A)\ e$.
- If the returned integer is pure, return it with the associated proof.
- Otherwise, return a dummy integer and failing proof.

Easy to show that `ip` is actually valid in $\mathcal{T}_{\mathbb{E}}^{p}$. $\square$

Another Result for Free

Recall Markov's principle:

$$\Pi P : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\Sigma n : \mathbb{N}.\, P\ n = \texttt{true}) \to \Sigma n : \mathbb{N}.\, P\ n = \texttt{true} \qquad \text{(MP)}$$

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Corollary

$\nvdash_{\mathcal{T}^p_{\mathbb{E}}} \mathrm{MP}$ and thus $\nvdash_{\mathrm{CIC}} \mathrm{MP}$.

(This was proved recently by Coquand-Mannaa, although in a completely different way.)

Function Intensionality

Another interesting consequence that is similar to what happens in KR.

- $\mathcal{T}_{\mathbb{E}}^{p}$ satisfies definitional η-expansion: $\lambda x : A.\, M\, x \equiv M$.
- But it violates function extensionality!

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The reason is that there are invalid proofs of $\mathbb{1}$.

You cannot build them, but they exists as phantom arguments.

What Else?

What kind of similar horrors can we do in $\mathcal{T}_{\mathbb{E}}^{p}$?

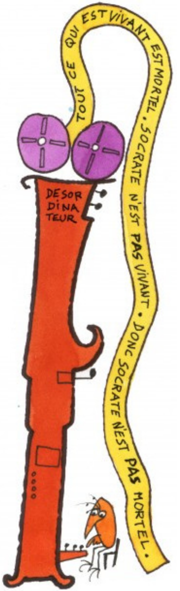
- I don't know!
- But there are probably lessons to be taken from realizability
- I'm probably pissing off both HoTT and PRL zealots by now

Get You A Larger Coq, Today!

We implemented $\mathcal{T}_{\mathbb{E}}$ and $\mathcal{T}_{\mathbb{E}}^{p}$ in Coq in a plugin.

`https://github.com/CoqHott/exceptional-tt`

- Allows to add exceptions to Coq just today.
- Compile effectful terms on the fly.
- Allows to reason about them in Coq.
- Write mind-blowing low-level code!



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"The more it fails, the more likely it will eventually succeed."

TODO When I Have a Permanent Position

- $\mathcal{T}_{\mathbb{E}}$ looks like a good intermediate language for model building
- The Calculus of Shadok Constructions
- Potential applications to Gradual Typing?
- Syntactic models are super cool! Let's write more!


C'EST TOUT POUR AUJOURD'HUI

Food For Thought

It seems you need to have a name starting with K to name a realizability.

Kleene
Kreisel
Krivine